

# Single-exterior Black Holes

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**Abstract.** We discuss quantum properties of the single-exterior, “geon”-type black (and white) holes that are obtained from the Kruskal spacetime and the spinless Bañados-Teitelboim-Zanelli hole via a quotient construction that identifies the two exterior regions. For the four-dimensional geon, the Hartle-Hawking type state of a massless scalar field is thermal in a limited sense, but there is a discrepancy between Lorentzian and Riemannian derivations of the geon entropy. For the three-dimensional geon, the state induced for a free conformal scalar field on the conformal boundary is similarly thermal in a limited sense, and the correlations in this state provide support for the holographic hypothesis in the context of asymptotically Anti-de Sitter black holes in string theory.<sup>1</sup>

## 1 Introduction

In quantum field theory on the Kruskal spacetime, one way to arrive at the thermal effects is through the observation that the spacetime has two exterior regions separated by a bifurcate Killing horizon. A free scalar field on the Kruskal spacetime has a vacuum state, known as the Hartle-Hawking vacuum [1,2], that is invariant under all the continuous isometries of the spacetime [3,4]. This state is pure, but the expectation values of operators with support in *one* exterior region are thermal in the Hawking temperature [1,2,3,4,5]. Similar observations hold for field theory on the nonextremal (2+1)-dimensional Bañados-Teitelboim-Zanelli (BTZ) black hole, both with and without spin [6], and also for conformal field theory on the conformal boundary of the BTZ hole [7,8,9].

In all these cases one has a vacuum state that knows about the global geometry of the spacetime, in particular about the fact that the spacetime has two exterior regions. Suppose now that we modify the spacetime in some

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‘reasonable’ fashion so that one exterior region remains as it is, and all the modification takes place behind the Killing horizons of this exterior region. Suppose further that the modified spacetime admits a vacuum state that is, in some reasonable sense, a Hartle-Hawking type vacuum. Can we then, by probing the new vacuum in the unmodified exterior region, discover that something has happened to the spacetime behind the horizons? In particular, as the new spacetime is still a black (and white) hole, does the new vacuum exhibit thermality, and if so, at what temperature? In the (2+1)-dimensional case, the analogous questions can also be raised for conformal field theory on the conformal boundary.

These lectures address the above questions for a particular modification of the Kruskal manifold and the spinless BTZ hole: we modify the spacetimes by a quotient construction that identifies the two exterior regions with each other. For Kruskal, the resulting spacetime is referred to as the  $\mathbb{RP}^3$  geon [10,11], and for BTZ, as the  $\mathbb{RP}^2$  geon [9]. These spacetimes are black (and white) holes, and their only singularities are those inherited from the singularities of the two-exterior holes.

On the  $\mathbb{RP}^3$  geon, a free scalar field has a vacuum induced from the Hartle-Hawking vacuum on Kruskal. The vacuum is not fully thermal for static exterior observers, but it appears thermal when probed with operators that do not see certain types of correlations, such as in particular operators with support at asymptotically late times, and the apparent temperature is then the usual Hawking temperature. However, a naive application of Euclidean-signature path-integral methods via saddle-point methods yields for the geon only half of the Bekenstein-Hawking entropy of the Schwarzschild hole with the same mass.

The situation on the conformal boundary of the  $\mathbb{RP}^2$  geon is analogous. The quotient construction from the conformal boundary of Anti-de Sitter space induces on the boundary of the geon a Hartle-Hawking type vacuum that is not fully thermal, but it appears thermal when probed with operators that do not see certain types of correlations, and the apparent temperature is then the usual Hawking temperature of the BTZ hole. The properties of the boundary vacuum turn out to reflect in a surprisingly close fashion the geometry of the geon spacetime. This can be interpreted as support for the holographic hypothesis [12,13], according to which physics in the bulk of a spacetime should be retrievable from physics on the boundary of the spacetime. It further suggests that single-exterior black holes can serve as a test bed for the versions of the holographic hypothesis that arise in string theory for asymptotically Anti-de Sitter spacetimes via the Maldacena duality conjectures [7,14,15,16].

The material is based on joint work [9,17] with Don Marolf, whom I would like to thank for a truly delightful collaboration. I would also like to thank the organizers of the Polanica Winter School for the opportunity to present the work in a most pleasant and inspiring atmosphere.

## 2 Kruskal Manifold and the $\mathbb{RP}^3$ Geon

Recall that the metric on the Kruskal manifold  $\mathcal{M}^L$  reads

$$ds^2 = \frac{32M^3}{r} \exp\left(-\frac{r}{2M}\right) (-dT^2 + dX^2) + r^2 d\Omega^2, \quad (1)$$

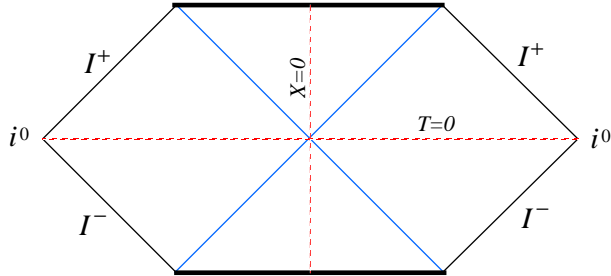
where  $d\Omega^2 = d\theta^2 + \sin^2\theta d\varphi^2$  is the metric on the unit two-sphere,  $M > 0$ ,  $X^2 - T^2 > -1$ , and  $r$  is determined as a function of  $T$  and  $X$  by

$$\left(\frac{r}{2M} - 1\right) \exp\left(\frac{r}{2M}\right) = X^2 - T^2. \quad (2)$$

The coordinates are global, apart from the elementary singularities of the spherical coordinates.  $\mathcal{M}^L$  is manifestly spherically symmetric, and it has in addition the Killing vector

$$V^L := \frac{1}{4M} (X\partial_T + T\partial_X), \quad (3)$$

which is timelike for  $|X| > |T|$  and spacelike for  $|X| < |T|$ . A conformal diagram of  $\mathcal{M}^L$ , with the two-spheres suppressed, is shown in Fig. 1.



**Fig. 1.** Conformal diagram of the Kruskal spacetime. Each point represents a suppressed  $S^2$  orbit of the  $O(3)$  isometry group

In each of the four quadrants of  $\mathcal{M}^L$  one can introduce Schwarzschild coordinates  $(t, r, \theta, \varphi)$  that are adapted to the isometry generated by  $V^L$ . In the “right-hand-side” exterior region,  $X > |T|$ , the coordinate transformation reads

$$\begin{aligned} T &= \left(\frac{r}{2M} - 1\right)^{1/2} \exp\left(\frac{r}{4M}\right) \sinh\left(\frac{t}{4M}\right), \\ X &= \left(\frac{r}{2M} - 1\right)^{1/2} \exp\left(\frac{r}{4M}\right) \cosh\left(\frac{t}{4M}\right), \end{aligned} \quad (4)$$

with  $r > 2M$  and  $-\infty < t < \infty$ . The exterior metric takes then the Schwarzschild form

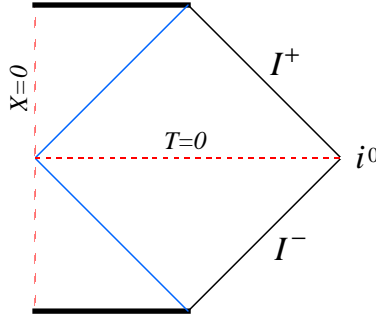
$$ds^2 = -\left(1 - \frac{r}{2M}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{r}{2M}\right)} + r^2 d\Omega^2, \quad (5)$$

and  $V^L = \partial_t$ .

Consider now on  $\mathcal{M}^L$  the isometry

$$J^L : (T, X, \theta, \varphi) \mapsto (T, -X, \pi - \theta, \varphi + \pi). \quad (6)$$

$J^L$  is clearly involutive, it acts properly discontinuously, it preserves the time orientation and spatial orientation, and it commutes with the spherical symmetry of  $\mathcal{M}^L$ . The quotient space  $\mathcal{M}^L/J^L$  is therefore a spherically symmetric, space and time orientable manifold. A conformal diagram of  $\mathcal{M}^L/J^L$  is shown in Fig. 2.  $\mathcal{M}^L/J^L$  is an inextendible black (and white) hole spacetime, and its only singularities are those inherited from the singularities of  $\mathcal{M}^L$ . It has only one exterior region, and its spatial topology is  $\mathbb{RP}^3 \setminus \{\text{point at infinity}\}$ . We refer to  $\mathcal{M}^L/J^L$  as the  $\mathbb{RP}^3$  geon [10,11].



**Fig. 2.** Conformal diagram of the  $\mathbb{RP}^3$  geon  $\mathcal{M}^L/J^L$ . Each point represents a suppressed orbit of the  $O(3)$  isometry group. The region  $X > 0$  is isometric to the region  $X > 0$  of  $\mathcal{M}^L$ , shown in Fig. 1, and the  $O(3)$  isometry orbits in this region are two-spheres. At  $X = 0$ , the  $O(3)$  orbits have topology  $\mathbb{RP}^2$ .

The exterior region of  $\mathcal{M}^L/J^L$  is clearly isometric to an exterior region of  $\mathcal{M}^L$ . In terms of the coordinates shown in Fig. 2, the exterior region is at  $X > |T|$ , and one can introduce in the exterior region standard Schwarzschild coordinates by (4). As the Killing vector  $V^L$  on  $\mathcal{M}^L$  changes its sign under  $J^L$ , the timelike Killing vector  $\partial_t$  on the exterior of  $\mathcal{M}^L/J^L$  can however not be continued into a globally-defined Killing vector on  $\mathcal{M}^L/J^L$ . This means that not all the constant  $t$  hypersurfaces in the exterior region of  $\mathcal{M}^L/J^L$  are equal: among them, there is only one (in Fig. 2, the one at

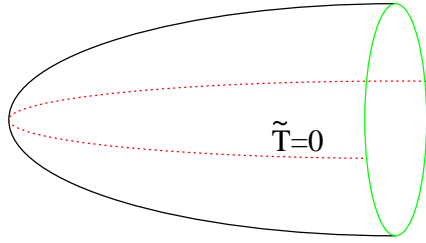
$T = 0$ ) that can be extended into a smoothly-embedded Cauchy hypersurface for  $\mathcal{M}^L/J^L$ .

The quotient construction from  $\mathcal{M}^L$  to  $\mathcal{M}^L/J^L$  can be analytically continued to the Riemannian (i.e., positive definite) sections via the formalism of (anti-)holomorphic involutions [18,19]. The Riemannian section of the Kruskal hole, denoted by  $\mathcal{M}^R$ , is obtained from (1) and (2) by setting  $T = -i\tilde{T}$  and letting  $\tilde{T}$  and  $X$  take all real values [20]. The analytic continuation of  $J^L$ , denoted by  $J^R$ , acts on  $\mathcal{M}^R$  by

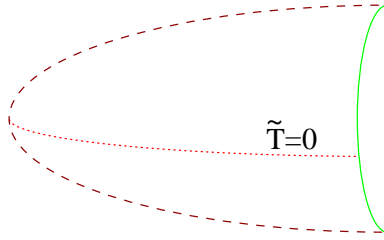
$$J^R : (\tilde{T}, X, \theta, \varphi) \mapsto (\tilde{T}, -X, \pi - \theta, \varphi + \pi), \quad (7)$$

and the Riemannian section of the  $\mathbb{RP}^3$  geon is  $\mathcal{M}^R/J^R$ .

On  $\mathcal{M}^R$  we can introduce the Riemannian Schwarzschild coordinates  $(\tilde{t}, r, \theta, \varphi)$ , obtained from the Lorentzian Schwarzschild coordinates for  $r > 2M$  by  $t = -i\tilde{t}$ . These Riemannian Schwarzschild coordinates are global, with the exception of a coordinate singularity at the Riemannian horizon  $r = 2M$ , provided they are understood with the identification  $(\tilde{t}, r, \theta, \varphi) \sim (\tilde{t} + 8\pi M, r, \theta, \varphi)$  [20]. On  $\mathcal{M}^R/J^R$ , the Riemannian Schwarzschild coordinates need to be understood with the additional identification  $(\tilde{t}, r, \theta, \varphi) \sim (\tilde{t} + 4\pi M, r, \pi - \theta, \varphi + \pi)$ , which arises from the action (7) of  $J^R$  on  $\mathcal{M}^R$ . The Killing vector  $\partial_{\tilde{t}}$  is global on  $\mathcal{M}^R$ , and it generates an  $U(1)$  isometry group with a fixed point at the Riemannian horizon. On  $\mathcal{M}^R/J^R$ , on the other hand,  $\partial_{\tilde{t}}$  is global only as a line field but not as a vector field, and the analogous  $U(1)$  isometry does not exist. Embedding diagrams of  $\mathcal{M}^R$  and  $\mathcal{M}^R/J^R$ , with the orbits of the spherical symmetry suppressed, are shown in Figs. 3 and 4.



**Fig. 3.** A “sock” representation of the Riemannian section  $\mathcal{M}^R$  of the complexified Kruskal manifold. The  $S^2$  orbits of the  $O(3)$  isometry group are suppressed, and the remaining two dimensions  $(\tilde{T}, X)$  are shown as an isometric embedding into Euclidean  $\mathbb{R}^3$ . The isometry generated by  $\partial_{\tilde{t}}$  rotates the two shown dimensions



**Fig. 4.** A representation of the Riemannian section  $\mathcal{M}^R/J^R$  of the complexified  $\mathbb{RP}^3$  geon as the “front half” of the the  $\mathcal{M}^R$  sock. The orbits of the  $O(3)$  isometry group are suppressed, as in Fig. 3. The generic orbits have topology  $S^2$ , but those at the “boundary” of the diagram (*dashed line*) have topology  $\mathbb{RP}^2$

### 3 Vacua on Kruskal and on the $\mathbb{RP}^3$ Geon

We now consider a free scalar field on the Kruskal manifold and on the  $\mathbb{RP}^3$  geon. For concreteness, we take here the field to be massless. The situation with a massive field is qualitatively similar [17].

Recall that the Hartle-Hawking vacuum  $|0_K\rangle$  of a massless scalar field on the Kruskal manifold  $\mathcal{M}^L$  can be characterized by its positive frequency properties along the affine parameters of the horizon generators [1,2,5], by the complex analytic properties of the Feynman propagator upon analytic continuation to  $\mathcal{M}^R$  [1], or by the invariance under the continuous isometries of  $\mathcal{M}^L$  [3,4].  $|0_K\rangle$  is regular everywhere on  $\mathcal{M}^L$ , but it is not annihilated by the annihilation operators associated with the future timelike Killing vectors in the exterior regions: a static observer in an exterior region sees  $|0_K\rangle$  as an excited state. We have the expansion

$$|0_K\rangle = \sum_{i \dots k} f_{i \dots k} (a_i^R)^\dagger (a_i^L)^\dagger \dots (a_k^R)^\dagger (a_k^L)^\dagger |0_{B,K}\rangle, \quad (8)$$

where the Boulware vacuum  $|0_{B,K}\rangle$  is the vacuum with respect to the timelike Killing vectors in the exterior regions,  $(a_i^R)^\dagger$  are the creation operators with respect to this Killing vector in the right-hand-side exterior region, and  $(a_i^L)^\dagger$  are the creation operators with respect to this Killing vector in the left-hand-side exterior region.

$|0_K\rangle$  thus contains Boulware excitations in correlated pairs, such that one member of the pair has support in the right-hand-side exterior and the other member in the left-hand-side exterior. An operator with support in (say) the right-hand-side exterior does not couple to the left-hand-side excitations, and the expectation values of such operators in  $|0_K\rangle$  thus look like expectation values in a mixed state. From the detailed form of the expansion coefficients

$f_{i\dots k}$  (which we do not write out here) it is seen that this mixed state is thermal, and it has at infinity the Hawking temperature  $T = (8\pi M)^{-1}$ .

Now, through the quotient construction from  $\mathcal{M}^L$  to  $\mathcal{M}^L/J^L$ ,  $|0_K\rangle$  induces on  $\mathcal{M}^L/J^L$  a Hartle-Hawking type vacuum, which we denote by  $|0_G\rangle$ . Again,  $|0_G\rangle$  can be characterized by its positive frequency properties along the affine parameters of the horizon generators, or by the complex analytic properties of the Feynman propagator [17].  $|0_G\rangle$  has the expansion

$$|0_G\rangle = \sum_{i\dots k} \tilde{f}_{i\dots k} \left(\tilde{a}_i^{(1)}\right)^\dagger \left(\tilde{a}_i^{(2)}\right)^\dagger \cdots \left(\tilde{a}_k^{(1)}\right)^\dagger \left(\tilde{a}_k^{(2)}\right)^\dagger |0_{B,G}\rangle, \quad (9)$$

where  $|0_{B,G}\rangle$  is the Boulware vacuum in the single exterior region and  $\left(\tilde{a}_i^{(\alpha)}\right)^\dagger$  are the creation operators of Boulware particles in the exterior region. The indices  $i$  and  $\alpha$  now label a complete set of positive frequency Boulware modes in the single exterior region.

We see from (9) that  $|0_G\rangle$  contains Boulware excitations in correlated pairs, but the crucial point is that both members of each pair have support in the single exterior region. Consequently, the expectation values of arbitrary operators in the exterior region are not thermal. However, for operators that do not contain couplings between modes with  $\alpha = 1$  and  $\alpha = 2$ , the expectation values turn out to be thermal, with the Hawking temperature  $T = (8\pi M)^{-1}$ . One class of operators for which this is the case are operators with, roughly speaking, support at asymptotically late (or early) times: the reason is that an excitation with support at asymptotically late exterior times is correlated with one with support at asymptotically early exterior times. Note that “early” and “late” here mean compared with the distinguished exterior spacelike hypersurface mentioned in Sect. 2 (in Fig. 2, the one at  $T = 0$ ).

Thus, for a late-time observer in the exterior region of  $\mathcal{M}^L/J^L$ , the state  $|0_G\rangle$  is indistinguishable from the state  $|0_K\rangle$  on  $\mathcal{M}^L$ . This conclusion can also be reached by analyzing the response of a monopole particle detector, or from an emission-absorption analysis analogous to that performed for  $|0_K\rangle$  in [1], provided certain technical assumptions about the falloff of the two-point functions in  $|0_G\rangle$  hold [17].

## 4 Entropy of the $\mathbb{RP}^3$ Geon?

As explained above, for a late-time exterior observer in  $\mathcal{M}^L/J^L$  the state  $|0_G\rangle$  is indistinguishable from the state  $|0_K\rangle$  on  $\mathcal{M}^L$ . The late-time observer can therefore promote the classical first law of black hole mechanics [21] into a first law of black hole thermodynamics exactly as for the Kruskal black hole [22,23,24]. The observer thus finds for the thermodynamic late time entropy of the geon the usual Kruskal value  $4\pi M^2$ , which is one quarter of the area of the geon black hole horizon at late times. If one views the geon

as a dynamical black-hole spacetime, with the asymptotic far-future horizon area  $16\pi M^2$ , this is the result one might have expected on physical grounds.

On the other hand, the area-entropy relation for the geon is made subtle by the fact that the horizon area is not constant along the horizon. Away from the intersection of the past and future horizons, the horizon duly has topology  $S^2$  and area  $16\pi M^2$ , just as in Kruskal. The critical surface at the intersection of the past and future horizons, however, has topology  $\mathbb{RP}^2$  and area  $8\pi M^2$ . As it is precisely this critical surface that belongs to both the Lorentzian and Riemannian sections of the complexified manifold, and constitutes the horizon of the Riemannian section, one may expect that methods utilizing the Riemannian section of the complexified manifold [20,25] produce for the geon entropy the value  $2\pi M^2$ , which is one quarter of the critical surface area, and only half of the Kruskal entropy. This indeed is the case, provided the surface terms in the Riemannian geon action are handled in a way suggested by the quotient construction from  $\mathcal{M}^R$  to  $\mathcal{M}^R/J^R$  [17].

There are several possible physical interpretations for this disagreement between the Lorentzian and Riemannian results for the entropy. At one extreme, it could be that the path-integral framework is simply inapplicable to the geon, for reasons having to do with the absence of certain globally-defined symmetries. For instance, despite the fact that the exterior region of  $\mathcal{M}^L/J^L$  is static, the restriction of  $|0_G\rangle$  to this region is not. Also, the asymptotic region of  $\mathcal{M}^R/J^R$  does not have a globally-defined Killing field, and the homotopy group of any neighborhood of infinity in  $\mathcal{M}^R/J^R$  is  $\mathbb{Z}_2$  as opposed to the trivial group. It may well be that such an asymptotic structure does not satisfy the boundary conditions that should be imposed in the path integral for the quantum gravitational partition function.

At another extreme, it could be that the path-integral framework is applicable to the geon, and our way of applying it is correct, but the resulting entropy is physically distinct from the subjective thermodynamic entropy associated with the late-time exterior observer. If this is the case, the physical interpretation of the path-integral entropy might be in the quantum statistics in the whole exterior region, and one might anticipate this entropy to arise from tracing over degrees of freedom that are in some sense unobservable. It would thus be interesting to see whether any state-counting calculation for the geon entropy would produce agreement with the path-integral result.

## 5 $\text{AdS}_3$ , the Spinless Nonextremal BTZ Hole, and the $\mathbb{RP}^2$ Geon

We now turn to 2+1 spacetime dimensions. In this section we review how the spinless nonextremal BTZ hole and the  $\mathbb{RP}^2$  geon arise as quotient spaces of the three-dimensional Anti-de Sitter space, and how this quotient construction can be extended to the conformal boundaries.



### 5.1 $\text{AdS}_3$ , its Covering Space, and the Conformal Boundary

Recall that the three-dimensional Anti-de Sitter space ( $\text{AdS}_3$ ) can be defined as the hyperboloid

$$-1 = -(T^1)^2 - (T^2)^2 + (X^1)^2 + (X^2)^2 \quad (10)$$

in  $\mathbb{R}^{2,2}$  with the metric

$$ds^2 = -(dT^1)^2 - (dT^2)^2 + (dX^1)^2 + (dX^2)^2. \quad (11)$$

We have here normalized the Gaussian curvature of  $\text{AdS}_3$  to  $-1$ . This embedding representation makes transparent the fact that  $\text{AdS}_3$  is a maximally symmetric space with the isometry group  $\text{O}(2,2)$ .

For understanding the structure of the infinity, we introduce the coordinates  $(t, \rho, \theta)$  by [26]

$$\begin{aligned} T^1 &= \frac{1+\rho^2}{1-\rho^2} \cos t, & T^2 &= \frac{1+\rho^2}{1-\rho^2} \sin t, \\ X^1 &= \frac{2\rho}{1-\rho^2} \cos \theta, & X^2 &= \frac{2\rho}{1-\rho^2} \sin \theta. \end{aligned} \quad (12)$$

With  $0 \leq \rho < 1$  and the identifications  $(t, \rho, \theta) \sim (t, \rho, \theta + 2\pi) \sim (t + 2\pi, \rho, \theta)$ , these coordinates can be understood as global on  $\text{AdS}_3$ , apart from the elementary coordinate singularity at  $\rho = 0$ . The metric reads

$$ds^2 = \frac{4}{(1-\rho^2)^2} \left[ -\frac{1}{4}(1+\rho^2)^2 dt^2 + d\rho^2 + \rho^2 d\theta^2 \right]. \quad (13)$$

Dropping now from (13) the conformal factor  $4(1-\rho^2)^{-2}$  yields a spacetime that can be regularly extended to  $\rho = 1$ , and the timelike hypersurface  $\rho = 1$  in this conformal spacetime is by definition the conformal boundary of  $\text{AdS}_3$ . It is a timelike two-torus coordinatized by  $(t, \theta)$  with the identifications  $(t, \theta) \sim (t, \theta + 2\pi) \sim (t + 2\pi, \theta)$ , and it has the flat metric

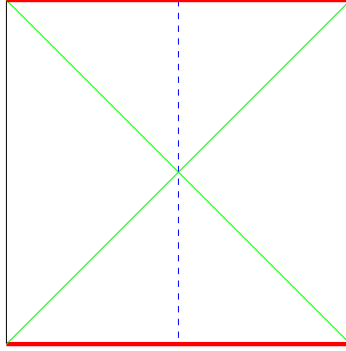
$$ds^2 = -dt^2 + d\theta^2. \quad (14)$$

The conformal boundary construction generalizes in an obvious way to the universal covering space of  $\text{AdS}_3$ , which we denote by  $\text{CAdS}_3$ . The only difference is that the coordinate  $t$  is not periodically identified. The conformal boundary of  $\text{CAdS}_3$ , which we denote by  $B_C$ , is thus a timelike cylinder with the metric (14) and the identification  $(t, \theta) \sim (t, \theta + 2\pi)$ .

### 5.2 The Spinless Nonextremal BTZ Hole

Let  $\xi_{\text{int}}$  be on  $\text{CAdS}_3$  the Killing vector induced by the boost-like Killing vector  $\xi_{\text{emb}} := -T^1 \partial_{X^1} - X^1 \partial_{T^1}$  of  $\mathbb{R}^{2,2}$ , and let  $D_{\text{int}}$  denote the largest subset

of  $\text{CAdS}_3$  that contains the hypersurface  $t = 0$  and in which  $\xi_{\text{int}}$  is spacelike. Given a prescribed positive parameter  $a$ , the isometry  $\exp(a\xi_{\text{int}})$  generates a discrete isometry group  $\Gamma_{\text{int}} \simeq \mathbb{Z}$  of  $D_{\text{int}}$ . The spinless nonextremal BTZ hole is by definition the quotient space  $D_{\text{int}}/\Gamma_{\text{int}}$  [6,27]. A conformal diagram, with the  $S^1$  factor arising from the identification suppressed, is shown in Fig. 5. The horizon circumference is  $a$ , and the ADM mass is  $M = a^2/(32\pi^2 G_3)$ , where  $G_3$  is the (2+1)-dimensional Newton's constant. For further discussion, including expressions for the metric in coordinates adapted to the isometries, we refer to [6,27].



**Fig. 5.** A conformal diagram of the BTZ hole. Each point in the diagram represents a suppressed  $S^1$ . The involution  $\tilde{J}_{\text{int}}$  introduced in Subsect. 5.3 consists of a left-right reflection about the dashed vertical line, followed by a rotation by  $\pi$  on the suppressed  $S^1$ .

As seen in Fig. 5, the BTZ hole has two exterior regions, and the infinities are asymptotically Anti-de Sitter. The point of interest for us is that each of the infinities has a conformal boundary that is induced from  $B_C$  by the quotient construction. Technically, one observes that  $\xi_{\text{int}}$  induces on  $B_C$  the conformal Killing vector  $\xi := \cos t \sin \theta \partial_\theta + \sin t \cos \theta \partial_t$ , and that  $D_{\text{int}}$  reaches  $B_C$  in the two diamonds

$$\begin{aligned} D_R &:= \{(t, \theta) \mid 0 < \theta < \pi, |t| < \pi/2 - |\theta - \pi/2|\} , \\ D_L &:= \{(t, \theta) \mid -\pi < \theta < 0, |t| < \pi/2 - |\theta + \pi/2|\} . \end{aligned} \quad (15)$$

The two conformal boundaries of the BTZ hole are then the quotient spaces  $D_R/\Gamma_R$  and  $D_L/\Gamma_L$ , where  $\Gamma_R$  and  $\Gamma_L$  are the restrictions to respectively  $D_R$  and  $D_L$  of the conformal isometry group of  $B_C$  generated by  $\exp(a\xi)$  [7,8]. To make this explicit, we cover  $D_R$  by the coordinates

$$\begin{aligned} \alpha &= -\ln \tan [(\theta - t)/2] , \\ \beta &= \ln \tan [(\theta + t)/2] , \end{aligned} \quad (16)$$

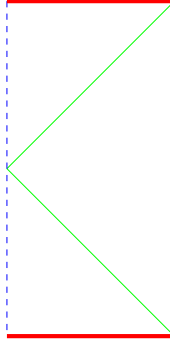
in which the metric induced from (14) is conformal to

$$ds^2 = -\left(\frac{2\pi}{a}\right)^2 d\alpha d\beta, \quad (17)$$

and  $\xi = -\partial_\alpha + \partial_\beta$ . The quotient space  $D_R/\Gamma_R$ , with the metric induced from (17), is thus isometric to  $B_C$  with the metric (14). In particular, it has topology  $\mathbb{R} \times S^1$ . It can be shown [7,8,9] that the conventionally-normalized Killing vector of the BTZ hole that is timelike in the exterior regions induces on  $D_R/\Gamma_R$  the timelike Killing vector  $\eta = \partial_\alpha + \partial_\beta$ . Analogous observations apply to  $D_L/\Gamma_L$ .

### 5.3 The $\mathbb{RP}^2$ Geon

The  $\mathbb{RP}^2$  geon is obtained from the spinless BTZ hole in close analogy with the quotient construction used with the  $\mathbb{RP}^3$  geon in Sect. 2. We denote the relevant involutive isometry of the BTZ hole by  $\tilde{J}_{\text{int}}$ : in the conformal diagram of Fig. 5,  $\tilde{J}_{\text{int}}$  consists of a left-right reflection about the dashed vertical line, followed by a rotation by  $\pi$  on the suppressed  $S^1$ . A conformal diagram of the quotient space, the  $\mathbb{RP}^2$  geon, is shown in Fig. 6. It is clear that the  $\mathbb{RP}^2$  geon is a black (and white) hole spacetime with a single exterior region that is isometric to one exterior region of the BTZ hole. It is time orientable but not space orientable, and the spatial topology is  $\mathbb{RP}^2 \setminus \{\text{point at infinity}\}$ . The local and global isometries closely parallel those of the  $\mathbb{RP}^3$  geon [9].



**Fig. 6.** A conformal diagram of the  $\mathbb{RP}^2$  geon. The region not on the dashed line is identical to that in the diagram of Fig. 5, each point representing a suppressed  $S^1$  in the spacetime. On the dashed line, each point in the diagram represents again an  $S^1$  in the spacetime, but with only half of the circumference of the  $S^1$ 's in the diagram of Fig. 5

The map  $\tilde{J}_{\text{int}}$  can clearly be extended to the conformal boundary of the BTZ hole, where it defines an involution  $\tilde{J}$  that interchanges the two bound-

ary components. Quotienting the conformal boundary of the BTZ hole by this involution gives the conformal boundary of the  $\mathbb{RP}^2$  geon, which is thus isomorphic to one boundary component of the BTZ hole. Note that although the  $\mathbb{RP}^2$  geon is not space orientable, its conformal boundary  $\mathbb{R} \times S^1$  is.

## 6 Vacua on the Conformal Boundaries

We now turn to a free conformal scalar field on the boundaries of  $\text{CAdS}_3$ , the BTZ hole, and the  $\mathbb{RP}^2$  geon.

Let  $|0\rangle$  denote on  $B_C$  the vacuum state with respect to the timelike Killing vector  $\partial_t$ . We wish to know what kind of states  $|0\rangle$  induces on the conformal boundaries of the BTZ hole and the  $\mathbb{RP}^2$  geon. For concreteness, we focus the presentation on the non-zero modes of the field. The subtleties with the zero-modes are discussed in [9].

Consider the boundary of the BTZ hole. As noted above, the timelike Killing vectors on the two components do not lift to the timelike Killing vector  $\partial_t$  on  $B_C$ : the future timelike Killing vector on  $D_R/\Gamma_R$  lifts to  $[a/(2\pi)]\eta$ , and an analogous statement holds for  $D_L/\Gamma_L$ . To interpret the state induced by  $|0\rangle$  on the BTZ hole boundary in terms of the BTZ particle modes, we must first write the state induced by  $|0\rangle$  on  $D_R \cup D_L$  in terms of continuum-normalized particle states that are positive frequency with respect to  $\eta$  on  $D_R$  and with respect to the analogous Killing vector on  $D_L$ , and then restrict to appropriately periodic field modes in order to accommodate the identification by  $\exp(a\xi)$ . This calculation is quite similar to expressing the Minkowski vacuum in terms of Rindler particle modes [4,28,29]. Denoting the state induced from  $|0\rangle$  by  $|\text{BTZ}\rangle$ , we have the expansion

$$|\text{BTZ}\rangle = \sum_{i \dots k} f_{i \dots k} (a_i^R)^\dagger (a_i^L)^\dagger \dots (a_k^R)^\dagger (a_k^L)^\dagger |0\rangle_R |0\rangle_L, \quad (18)$$

where  $|0\rangle_R$  and  $|0\rangle_L$  are respectively the vacua on the two boundary components with respect to their timelike Killing vectors,  $(a_i^R)^\dagger$  are the creation operators with respect to this Killing vector on  $D_R/\Gamma_R$ , and  $(a_i^L)^\dagger$  are the creation operators with respect to this Killing vector on  $D_L/\Gamma_L$ . The analogy to the expansion (8) of the Hartle-Hawking vacuum on Kruskal is clear: the excitations come in correlated pairs, the two members of each pair now living on different boundary components. Restriction to one boundary component yields a thermal state, and when the normalization of the boundary timelike Killing vector is matched to that in the bulk of the spacetime, the temperature turns out to be the Hawking temperature of the BTZ hole,  $a/4\pi^2$ . This is the result first found in [7].

The boundary of the  $\mathbb{RP}^2$  geon has a single connected component. Denoting the state induced from  $|0\rangle$  by  $|\mathbb{RP}^2\rangle$ , we have the expansion

$$|\mathbb{RP}^2\rangle = \sum_{i \dots k} \tilde{f}_{i \dots k} (\tilde{a}_i^{(+)})^\dagger (\tilde{a}_i^{(-)})^\dagger \dots (\tilde{a}_k^{(+)})^\dagger (\tilde{a}_k^{(-)})^\dagger |0\rangle_R, \quad (19)$$

where  $|0\rangle_R$  now denotes the geon boundary vacuum with respect to the time-like Killing vector,  $\left(\tilde{a}_i^{(\alpha)}\right)^\dagger$  are the creation operators with respect to this Killing vector, and the indices  $i$  and  $\alpha$  label the modes. The modes with  $\alpha = +$  are right-movers and the modes with  $\alpha = -$  are left-movers. The analogy to the expansion (9) of the Hartle-Hawking type vacuum on the  $\mathbb{RP}^3$  geon is clear. For operators that do not contain couplings between modes with  $\alpha = +$  and  $\alpha = -$ , the expectation values turn out to be thermal, with the BTZ Hawking temperature  $a/4\pi^2$ .

As shown in Table 1, several properties of the state  $|\mathbb{RP}^2\rangle$  reflect properties of the  $\mathbb{RP}^2$  geon spacetime geometry. First,  $|\mathbb{RP}^2\rangle$  is a *pure* state on the boundary cylinder  $\mathbb{R} \times S^1$ : this follows by construction since (unlike with the BTZ hole) the single cylinder constitutes the whole conformal boundary. Second,  $|\mathbb{RP}^2\rangle$  is an excited state with respect to the boundary timelike Killing field. This can be understood to reflect the fact that the spacetime attached to the boundary is not CAdS<sub>3</sub>. Third, it can be shown that  $|\mathbb{RP}^2\rangle$  is not invariant under translations generated by the timelike Killing vector on the boundary. This reflects the absence on the spacetime of a globally-defined Killing vector that would be timelike in the exterior region (cf. the discussion of the isometries of the  $\mathbb{RP}^3$  geon in Sect. 2). Thus,  $|\mathbb{RP}^2\rangle$  “knows” not just about the exterior region of the  $\mathbb{RP}^2$  geon but also about the region behind the horizons.

Fourth, the correlations in  $|\mathbb{RP}^2\rangle$  are between the right-movers and the left-movers. This is a direct consequence of the fact that the map  $\tilde{J}_{\text{int}}$  on the BTZ hole reverses the spatial orientation, and it reflects thus the spatial nonorientability of the geon. Fifth,  $|\mathbb{RP}^2\rangle$  appears thermal in the Hawking temperature for operators that do not see the correlations: this reflects the fact that the geon is a black (and white) hole spacetime.

Finally, the expectation value of the energy in  $|\mathbb{RP}^2\rangle$  is, in the limit  $a \gg 1$ , equal to  $a^2/48\pi^2$ , which is quadratic in  $a$  and thus proportional to the ADM mass of the geon. The energy expectation value in the state  $|\text{BTZ}\rangle$  on one boundary cylinder of the BTZ hole is also equal to  $a^2/48\pi^2$ , for  $a \gg 1$ . In this sense, the energy expectation value on a single boundary component is the same in  $|0\rangle_{\mathbb{RP}^2}$  and  $|\text{BTZ}\rangle$ . The analogous property in the spacetime is that the ADM mass at one infinity is not sensitive to whether a second infinity exists behind the horizons.

## 7 Holography and String Theory

We have seen that the state  $|\mathbb{RP}^2\rangle$  on the boundary cylinder of the  $\mathbb{RP}^2$  geon mirrors several aspects of the spacetime geometry of the  $\mathbb{RP}^2$  geon. Some of this mirroring is immediate from the construction, such as the property that  $|\mathbb{RP}^2\rangle$  is a pure state. Some aspects of the mirroring appear however quite nontrivial, especially the fact that the energy expectation value turned out to

**Table 1.** Properties of the state  $|\mathbb{RP}^2\rangle$ , and the corresponding properties of the  $\mathbb{RP}^2$  geon spacetime

$ \mathbb{RP}^2\rangle$	$\mathbb{RP}^2$ geon geometry
pure state	boundary connected
excited state	not Anti-de Sitter
not static	no global KVF
correlations: left-movers with right-movers	spatially nonorientable
right-movers (left-movers) thermal, $T = a/4\pi^2$	black hole, $T_H = a/4\pi^2$
$\langle E \rangle = a^2/48\pi^2$ , $a \gg 1$	$M = a^2/(32\pi^2 G_3)$

be proportional to the ADM mass, and with the same constant of proportionality as for the vacuum  $|\text{BTZ}\rangle$  on the boundary of the BTZ hole. One can see this as a piece of evidence in support of the holographic hypothesis [12,13], according to which physics in the bulk of a spacetime should be retrievable from physics on the boundary of the spacetime.

One would certainly not expect a free conformal scalar field on the boundary of a spacetime to carry all the information about the spacetime geometry. However, for certain spacetimes related to Anti-de Sitter space, a more precise version of the holographic hypothesis has emerged in string theory in the form of the Maldacena duality conjectures [7,14,15,16]. In particular, the 10-dimensional spacetime  $\text{CAdS}_3 \times S^3 \times T^4$ , with a flat metric on the  $T^4$  and a round metric on the  $S^3$ , is a classical solution to string theory, and the duality conjectures relate string theory on this spacetime to a certain conformal nonlinear sigma-model on the conformal boundary of the  $\text{CAdS}_3$  component. Upon quotienting from  $\text{CAdS}_3$  to the (in general spinning) BTZ hole, the conformal field theory on the boundary ends in a thermal state analogous to our  $|\text{BTZ}\rangle$ , but with an energy expectation value that in the high temperature limit is not merely proportional to but in fact equal to the ADM mass of the hole [7]. This result can be considered a strong piece of evidence for the duality conjectures.

It would now be of obvious interest to adapt our free scalar field analysis on the boundary of the  $\mathbb{RP}^2$  geon to a string theoretic context in which the duality conjectures would apply. One would expect the boundary state again to appear thermal in the Hawking temperature under some restricted set of observations. The crucial question for the the holographic hypothesis is how the correlations in the boundary state might reflect the geometry of the spacetime. As a preliminary step in this direction, a toy conformal field theory that mimics some of the anticipated features of the extra dimensions

was considered in [9], and the energy expectation value in this toy theory was found to be equal to the geon ADM mass in the high temperature limit.

## 8 Concluding Remarks

The results presented here for the  $\mathbb{RP}^3$  geon and the  $\mathbb{RP}^2$  geon provide evidence that single-exterior black holes offer a nontrivial arena for scrutinizing quantum physics of black holes. It remains a subject to future work to understand to what extent the results reflect the peculiarities of these particular spacetimes, and to what extent they might have broader validity.

In some respects the  $\mathbb{RP}^3$  and  $\mathbb{RP}^2$  geons are certainly quite nongeneric black hole spacetimes. For example, our quotient constructions on Kruskal and the spinless BTZ hole do not immediately generalize to accommodate spin, as the putative isometry would need to invert the angular momentum. Similarly, the quotient construction on Kruskal does not immediately generalize to the Reissner-Nordström hole, as the relevant isometry would invert the electric field. Also, the spatial nonorientability of the  $\mathbb{RP}^2$  geon may lead to difficulties in the string theoretic context. However, in 2+1 dimensions there exist locally Anti-de Sitter single-exterior black (and white) hole spacetimes that admit a spin, and one can choose their spatial topology to be orientable, for example  $T^2 \setminus \{\text{point at infinity}\}$  [26,30]. A natural next step would be to consider quantum field theory on these spinning “wormhole” spacetimes and on their conformal boundaries.

## References

1. Hartle J.B., Hawking S.W. (1976) Phys. Rev. D **13**, 2188
2. Israel W. (1976) Phys. Lett. **57A**, 107
3. Kay B.S., Wald R.M. (1991) Phys. Rep. **207**, 49
4. Wald R.M. (1994) Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics. The University of Chicago Press, Chicago
5. Unruh W.G. (1976) Phys. Rev. D **14**, 870
6. Carlip S. (1995) Class. Quantum Grav. **12**, 283 [gr-qc/9506079]
7. Maldacena J., Strominger A. (1998) J. High Energy Phys. **9812**, 005 [hep-th/9804085]
8. Horowitz G.T., Marolf D. (1998) J. High Energy Phys. **9807**, 014 [gr-qc/9805207]
9. Louko J., Marolf D. (1999) Phys. Rev. D **59**, 066002 [hep-th/9808081]
10. Giulini D. (1989) PhD Thesis, University of Cambridge, Cambridge
11. Friedman J.L., Schleich K., Witt D.M. (1993) Phys. Rev. Lett. **71**, 1486; Erratum (1995) Phys. Rev. Lett. **75**, 1872 [gr-qc/9305017]
12. 't Hooft G. (1993) In: Ali A., Ellis J., Randjbar-Daemi S. (Eds.) Salam-festschrift: A Collection of Talks. World Scientific, Singapore [gr-qc/932102]
13. Susskind L. (1995) J. Math. Phys. **36** 6377 [hep-th/9409089]
14. Maldacena J. (1997) Adv. Theor. Math. Phys. **2**, 231 [hep-th/9711200]

15. Gubser S.S., Klebanov I.R., Polyakov A.M. (1998) Phys. Lett. **B428** 105 [hep-th/9802109]; Witten E. (1998) Adv. Theor. Math. Phys. **2**, 253 [hep-th/9802150]; Claus P., Kallosh R., Kumar J., Townsend P., Van Proeyen A. (1998) J. High Energy Phys. **9806**, 004 [hep-th/9801206]; Susskind L., Witten E. (1998) e-print hep-th/9805114
16. Balasubramanian V., Kraus P., Lawrence A., Trivedi S.P. (1998) e-print hep-th/9808017; Keski-Vakkuri E. (1999) Phys. Rev. D **59** 104001 [hep-th/9808037]; Danielsson U.H., Keski-Vakkuri E., Kruczenski M. (1999) J. High Energy Phys. **9901**, 002 [hep-th/9812007]; Balasubramanian V., Giddings S.B., Lawrence A. (1999) J. High Energy Phys. **9903**, 001 [hep-th/9902052]
17. Louko J., Marolf D. (1998) Phys. Rev. D **58**, 024007 [gr-qc/9802068]
18. Gibbons G.W. (1992) In: Kim J.E. (Ed.) Proceedings of the 10th Sorak School of Theoretical Physics. World Scientific, Singapore
19. Chamblin A., Gibbons G.W. (1996) Phys. Rev. D **55**, 2177 [gr-qc/9607079]
20. Gibbons G.W., Hawking S.W. (1977) Phys. Rev. D **15**, 2752
21. Bardeen J.M., Carter B., Hawking S.W. (1973) Commun. Math. Phys. **31**, 161
22. Hawking S.W. (1975) Commun. Math. Phys. **43**, 199
23. Bekenstein J.D. (1972) Nuovo Cimento Lett. **4**, 737
24. Bekenstein J.D. (1974) Phys. Rev. D **9**, 3292
25. Hawking S.W. (1979) In: Hawking S.W., Israel W. (Eds.) General Relativity: An Einstein Centenary Survey. Cambridge University Press, Cambridge
26. Āminneborg S., Bengtsson I., Brill D.R., Holst S., Peldán P. (1998) Class. Quantum Grav. **15**, 627 [gr-qc/9707036]
27. Bañados M., Teitelboim C., Zanelli J. (1992) Phys. Rev. Lett. **69**, 1849 [hep-th/9204099]; Bañados M., Henneaux M., Teitelboim C., Zanelli J. (1993) Phys. Rev. D **48**, 1506 [gr-qc/9302012]
28. Birrell N.D., Davies P.C.W. (1982) Quantum Fields in Curved Space. Cambridge University Press, Cambridge
29. Takagi S. (1986) Prog. Theor. Phys. Suppl. **88**, 1
30. Āminneborg S., Bengtsson I., Holst S. (1999) Class. Quantum Grav. **16**, 363 [gr-qc/9805028]